Reduced Dynamics from the Unitary Group to Some Flag Manifolds : Interacting Matrix Riccati Equations

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Abstract

In this paper we treat the time evolution of unitary elements in the N level system and consider the reduced dynamics from the unitary group $U(N)$ to flag manifolds of the second type (in our terminology). Then we derive a set of differential equations of matrix Riccati types interacting with one another and present an important problem on a nonlinear superposition formula that the Riccati equation satisfies.

Our result is a natural generalization of the paper **Chaturvedi et al** (arXiv : 0706.0964 [quant-ph]).

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1 Introduction

In this paper we treat a finite quantum system (N–level system) and consider its unitary evolution in detail.

The unitary evolution for a time–dependent Hamiltonian $H(t)$ (which is not so artificial) is given by

$$i\hbar\frac{d}{dt}U(t) = H(t)U(t), \quad U(0) = E_N$$

where $H(t) \in H(N, \mathbf{C})$, $U(t) \in U(N)$ and E_N is the identity matrix. In the following we set $\hbar = 1$ and write

$$i\dot{U}(t) = H(t)U(t), \quad U(0) = E_N \tag{1}$$

for simplicity.

Now, we consider a reduction of symmetry. Namely, symmetry happens to reduce from the unitary group $U(N)$ to a subgroup H. Then the evolution on $U(N)$ "split" into evolutions on the subgroup H and corresponding homogeneous space $U(N)/H$.

We are interested in the evolution on $U(N)/H$ with special H and call this the reduced dynamics from $U(N)$ to $U(N)/H$ for short.

Especially, for $H = U(m) \times U(n)$ $(m+n = N)$ the homogeneous space is the Grassmann manifold and the matrix Riccati equation appears naturally, see [1] and [2].

Interesting enough, we meet the matrix Riccati equation(s) in several fields in Mathematics or Physics. In the special case $m = n = 1$ we have usual (complex–valued) Riccati equation. This equation has a mysterious formula called a nonlinear superposition one, see the text in detail.

In the paper we generalize the Grassmann manifold to some Flag manifold. Namely, for $H = U(l) \times U(m) \times U(n)$ $(l + m + n = N)$ we call $U(N)/H$ the flag manifold of the second type (temporarily). Then a set of differential equations of matrix Riccati types interacting with one another are obtained. This is the main result and we also present a problem on a nonlinear superposition when $l = m = n = 1$. See [3] as a general introduction to this topic.

As a result it may be concluded that "Riccati structure" appears naturally in the process

of symmetry reduction from the unitary group to some subgroup.

2 Reduced Dynamics on Grassmann Manifolds : Review

We set $m+n=N$ $(m, n \in \mathbf{N})$ and review some results from [1], [2] in the case of Grassmann mnnifolds. As a whole introduction in this section [4] is recommended.

The Grassmann manifold is the set of complex vector spaces defined by

$$G_{m,n}(\mathbf{C}) = \{\mathcal{V} \subset \mathbf{C}^{m+n} \mid \dim_{\mathbf{C}}\mathcal{V} = m\}. \tag{2}$$

Then it is well–known that

$$G_{m,n}(\mathbf{C}) \cong U(m+n)/U(m) \times U(n) \tag{3}$$

and moreover

$$U(m+n)/U(m) \times U(n) \cong GL(m+n)/B_{+} \tag{4}$$

where B_{+} is the (upper) Borel subgroup of $GL(m+n)$ given by

$$B_{+} = \left\{ \begin{pmatrix} *_1 & * \\ \mathbf{0} & *_2 \end{pmatrix} \in GL(m+n) \mid *_1 \in GL(m;\mathbf{C}),\ *_2 \in GL(n;\mathbf{C}) \right\}.$$

In order to obtain the element of $U(m+n)/U(m)\times U(n)$ from an element in $GL(m+n)/B_{+}$ it is convenient to use the orthonormalization (method) by Gram–Schmidt.

For the matrix

$$G \equiv \begin{pmatrix} E_m & \mathbf{0} \\ Z & E_n \end{pmatrix} \in GL(m+n)/B_{+} \tag{5}$$

where $Z \in M(n, m; \mathbf{C})$ we set

$$V_1 = \begin{pmatrix} E_m \\ Z \end{pmatrix}, \quad V_2 = \begin{pmatrix} \mathbf{0} \\ E_n \end{pmatrix}.$$

For $\{V_1, V_2\}$ the Gramm–Schmidt orthonormalization in the matrix form reads

$$\begin{aligned}
\hat{V}_1 &= V_1(V_1^\dagger V_1)^{-1/2} \Longrightarrow P_1 = \hat{V}_1\hat{V}_1^\dagger \ : \ \text{projection} \\
\tilde{V}_2 &= (E_{m+n} - P_1)V_2, \quad \hat{V}_2 = \tilde{V}_2(\tilde{V}_2^\dagger \tilde{V}_2)^{-1/2}.
\end{aligned}$$

Explicitly,

$$\hat{V}_1 = \begin{pmatrix} E_m \\ Z \end{pmatrix} L_Z^{-1/2}, \quad \hat{V}_2 = \begin{pmatrix} -Z^\dagger \\ E_n \end{pmatrix} M_Z^{-1/2}$$

where

$$L_Z = E_m + Z^\dagger Z, \quad M_Z = E_n + ZZ^\dagger.$$

Therefore we obtain the unitary matrix

$$\hat{V} = (\hat{V}_1, \hat{V}_2) = \begin{pmatrix} E_m & -Z^\dagger \\ Z & E_n \end{pmatrix} \begin{pmatrix} L_Z^{-1/2} & \\ & M_Z^{-1/2} \end{pmatrix} \in U(m+n). \tag{6}$$

Next, if we consider a transformation

$$G \longrightarrow G \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \in U(m) \times U(n)$$

then the resultant unitary matrix $\hat{V}$ also transforms like

$$\hat{V} \longrightarrow \hat{V} \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix}.$$

As a result, the procedure to obtain unitary matrices is **covariant** under the subgroup $U(m) \times U(n)$, so we can consider $\hat{V} \in G_{m,n}$.

A comment is in order. As another definition of the Grassmann manifold the following one in terms of projections is also well–known. See [5] and [6] as an elementery introduction and [7] as an advanced one.

$$\{P \in M(m+n; \mathbf{C}) \mid P^2 = P, \ P^\dagger = P, \ \mathrm{tr}P = m\} \cong U(m+n)/U(m) \times U(n).$$

The correspondence ($\hat{V} \longrightarrow P$) is as follows :

$$P = \hat{V} \begin{pmatrix} E_m & \\ & \mathbf{0} \end{pmatrix} \hat{V}^{-1} = \begin{pmatrix} E_m & -Z^\dagger \\ Z & E_n \end{pmatrix} \begin{pmatrix} E_m & \\ & \mathbf{0} \end{pmatrix} \begin{pmatrix} E_m & -Z^\dagger \\ Z & E_n \end{pmatrix}^{-1}.$$

Next, we treat the time–dependent Hamiltonian (which is not so artificial) like

$$H = H(t) = \begin{pmatrix} H_1(t) & V^\dagger(t) \\ V(t) & H_2(t) \end{pmatrix} \in \ H(m+n; \mathbf{C}). \tag{7}$$

Then the (reduced) evolution equation

$$i\frac{d}{dt}\hat{V} = H(t)\hat{V} \quad \text{where} \quad \hat{V} = \hat{V}(t) \iff Z = Z(t) \tag{8}$$

reduces to a matrix Riccati equation

$$i\dot{Z} = V + H_2 Z - Z H_1 - Z V^\dagger Z, \tag{9}$$

see [1], [2]. Note that we ignored the time evolution on U_1 and U_2, which is out of interest at the present time.

Especially, in the case of $m = n = 1$

$$\begin{aligned} H &= \begin{pmatrix} h_1(t) & \bar{v}(t) \\ v(t) & h_2(t) \end{pmatrix} && (10) \\ \hat{V} &= \frac{1}{\sqrt{1+|z(t)|^2}} \begin{pmatrix} 1 & -\bar{z}(t) \\ z(t) & 1 \end{pmatrix} && (11) \end{aligned}$$

the (matrix) Riccati equation becomes

$$i\dot{z} = v + (h_2 - h_1)z - \bar{v}z^2. \tag{12}$$

It is very interesting to note that this equation satisfies a mysterious formula called a **nonlinear superposition**. See for example [8] and its references. As a whole introduction to the Riccati equation see [9].

Namely, let $z_1,\ z_2,\ z_3$ be three different solutions and z be any solution. For the cross–ratio defined by

$$(z, z_1, z_2, z_3) \equiv \frac{z - z_1}{z - z_3} \div \frac{z_2 - z_1}{z_2 - z_3} \tag{13}$$

it is easy to see

$$i\frac{d}{dt}(z, z_1, z_2, z_3) = 0 \implies (z, z_1, z_2, z_3) = k$$

with constant k ($\in \mathbf{C}$). From this we can express z in terms of k and three solutions $z_1,\ z_2,\ z_3$ like

$$z = \frac{kz_3(z_2 - z_1) - z_1(z_2 - z_3)}{k(z_2 - z_1) - (z_2 - z_3)}. \tag{14}$$

This is called the nonlinear superposition formula for the Riccati equation. Concerning this formula we have the following

Problem Can this nonlinear superposition formula be derived (or reduced) from usual one in the linear equation (8) ?

3 Reduced Dynamics on Flag Manifolds

In this section we generalize the results in the preceding section based on Grassmann manifolds to Flag manifolds of the second type (in our terminology). See [10] and [11] as a good introduction.

For $l + m + n = N$ $(l, m, n \in \mathbf{N})$ the flag manifold of the second type is the sequence of complex vector spaces defined by

$$F_{l,m,n}(\mathbf{C}) = \{\mathcal{V} \subset \mathcal{W} \subset \mathbf{C}^{l+m+n} \mid \dim_{\mathbf{C}}\mathcal{V} = l,\ \dim_{\mathbf{C}}\mathcal{W} = l + m\}. \tag{15}$$

Then it is well–known that

$$F_{l,m,n}(\mathbf{C}) \cong U(l + m + n)/U(l) \times U(m) \times U(n) \tag{16}$$

and moreover

$$U(l + m + n)/U(l) \times U(m) \times U(n) \cong GL(l + m + n)/B_+ \tag{17}$$

where B_+ is the Borel subgroup given by

$$B_+ = \left\{ \begin{pmatrix} *_1 & * & * \\ \mathbf{0} & *_2 & * \\ \mathbf{0} & \mathbf{0} & *_3 \end{pmatrix} \in GL(l+m+n) \mid *_1 \in GL(l;\mathbf{C}),\ *_2 \in GL(m;\mathbf{C}),\ *_3 \in GL(n;\mathbf{C}) \right\}.$$

Similarly in the preceding section we consider the matrix

$$F \equiv \begin{pmatrix} E_l & \mathbf{0} & \mathbf{0} \\ X & E_m & \mathbf{0} \\ Y & Z & E_n \end{pmatrix} \in\ GL(l+m+n)/B_+ \tag{18}$$

and set

$$V_1 = \begin{pmatrix} E_l \\ X \\ Y \end{pmatrix}, \quad V_2 = \begin{pmatrix} \mathbf{0} \\ E_m \\ Z \end{pmatrix}, \quad V_3 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ E_n \end{pmatrix}.$$

For $\{V_1, V_2, V_3\}$ we perform the Gramm–Schmidt orthogonalization in the matrix form like

$$\begin{aligned}
\hat{V}_1 &= V_1(V_1^\dagger V_1)^{-1/2} \implies P_1 = \hat{V}_1\hat{V}_1^\dagger \ :\ \text{projection} \\
\tilde{V}_2 &= (E_{l+m+n} - P_1)V_2, \quad \hat{V}_2 = \tilde{V}_2(\tilde{V}_2^\dagger\tilde{V}_2)^{-1/2} \implies P_2 = \hat{V}_2\hat{V}_2^\dagger \ :\ \text{projection} \\
\tilde{V}_3 &= (E_{l+m+n} - P_1 - P_2)V_3 = (E_{l+m+n} - P_1)(E_{l+m+n} - P_2)V_3, \quad \hat{V}_3 = \tilde{V}_3(\tilde{V}_3^\dagger\tilde{V}_3)^{-1/2}.
\end{aligned}$$

Explicitly,

$$
\begin{aligned}
\hat{V}_1 &= \begin{pmatrix} E_l \\ X \\ Y \end{pmatrix} \Lambda^{-1/2}, \\
\hat{V}_2 &= \begin{pmatrix} -\Lambda^{-1}\delta^\dagger \\ E_m - X\Lambda^{-1}\delta^\dagger \\ Z - Y\Lambda^{-1}\delta^\dagger \end{pmatrix} \left(L - \delta\Lambda^{-1}\delta^\dagger\right)^{-1/2}, \\
\hat{V}_3 &= \begin{pmatrix} -\Lambda^{-1}Y^\dagger + \Lambda^{-1}\delta^\dagger \left(L - \delta\Lambda^{-1}\delta^\dagger\right)^{-1} (Z - Y\Lambda^{-1}\delta^\dagger)^\dagger \\ -X\Lambda^{-1}Y^\dagger - (E_m - X\Lambda^{-1}\delta^\dagger) \left(L - \delta\Lambda^{-1}\delta^\dagger\right)^{-1} (Z - Y\Lambda^{-1}\delta^\dagger)^\dagger \\ E_n - Y\Lambda^{-1}Y^\dagger - (Z - Y\Lambda^{-1}\delta^\dagger) \left(L - \delta\Lambda^{-1}\delta^\dagger\right)^{-1} (Z - Y\Lambda^{-1}\delta^\dagger)^\dagger \end{pmatrix} \times \\
& \quad \left(E_n - Y\Lambda^{-1}Y^\dagger - (Z - Y\Lambda^{-1}\delta^\dagger) \left(L - \delta\Lambda^{-1}\delta^\dagger\right)^{-1} (Z - Y\Lambda^{-1}\delta^\dagger)^\dagger\right)^{-1/2}
\end{aligned}
$$

where

$$\Lambda = E_l + X^\dagger X + Y^\dagger Y, \quad L = E_m + Z^\dagger Z, \quad \delta = X + Z^\dagger Y.$$

Therefore we obtain the unitary matrix

$$\hat{V} = (\hat{V}_1, \hat{V}_2, \hat{V}_3) \;\in\; U(l+m+n). \tag{19}$$

Here we list a decomposition of V

$$
\begin{aligned}
&\hat{V} = \\
&\begin{pmatrix} E_l & \mathbf{0} & \mathbf{0} \\ X & E_m & \mathbf{0} \\ Y & Z & E_n \end{pmatrix} \times \\
&\begin{pmatrix} E_l & -\Lambda^{-1}\delta^\dagger & -\Lambda^{-1}Y^\dagger + \Lambda^{-1}\delta^\dagger(L - \delta\Lambda^{-1}\delta^\dagger)^{-1}(Z - Y\Lambda^{-1}\delta^\dagger)^\dagger \\ \mathbf{0} & E_m & -(L - \delta\Lambda^{-1}\delta^\dagger)^{-1}(Z - Y\Lambda^{-1}\delta^\dagger)^\dagger \\ \mathbf{0} & \mathbf{0} & E_n \end{pmatrix} \times \\
&\begin{pmatrix} \Lambda^{-1/2} & & \\ & (L - \delta\Lambda^{-1}\delta^\dagger)^{-1/2} & \\ & & (E_n - Y\Lambda^{-1}Y^\dagger - (Z - Y\Lambda^{-1}\delta^\dagger) \left(L - \delta\Lambda^{-1}\delta^\dagger\right)^{-1} (Z - Y\Lambda^{-1}\delta^\dagger)^\dagger)^{-1/2} \end{pmatrix}.
\end{aligned}
$$

Compare $\hat{V}$ with the corresponding unitary matrix in [10] and [11] where $l = m = n = 1$.

If we consider a transformation

$$F \longrightarrow F \begin{pmatrix} U_1 & & \\ & U_2 & \\ & & U_3 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} U_1 & & \\ & U_2 & \\ & & U_3 \end{pmatrix} \in U(l) \times U(m) \times U(n)$$

then the resultant unitary matrix $\hat{V}$ also transforms like

$$\hat{V} \longrightarrow \hat{V} \begin{pmatrix} U_1 & & \\ & U_2 & \\ & & U_3 \end{pmatrix}.$$

As a result, the procedure to obtain unitary matrices is covariant under the subgroup $U(l) \times U(m) \times U(n)$, so we can consider $\hat{V} \in F_{l,m,n}$.

A comment is in order. As another definition of the flag manifold the following one in terms of projections is also well–known.

$$\{(P, Q) : \text{a pair of projections in } M(l+m+n; \mathbf{C}) \mid \mathrm{tr}P = l,\ \mathrm{tr}Q = l+m,\ PQ = P\}$$
$$\cong U(l+m+n)/U(l) \times U(m) \times U(n).$$

The correspondence $(\hat{V} \longrightarrow (P, Q))$ is as follows :

$$\begin{aligned} P &= \hat{V} \begin{pmatrix} E_l & & \\ & \mathbf{0} & \\ & & \mathbf{0} \end{pmatrix} \hat{V}^{-1} = W \begin{pmatrix} E_l & & \\ & \mathbf{0} & \\ & & \mathbf{0} \end{pmatrix} W^{-1}, \\ Q &= \hat{V} \begin{pmatrix} E_l & & \\ & E_m & \\ & & \mathbf{0} \end{pmatrix} \hat{V}^{-1} = W \begin{pmatrix} E_l & & \\ & E_m & \\ & & \mathbf{0} \end{pmatrix} W^{-1} \end{aligned}$$

where

$$W = \begin{pmatrix} E_l & -\Lambda^{-1}\delta^\dagger & -\Lambda^{-1}Y^\dagger + \Lambda^{-1}\delta^\dagger \left(L-\delta\Lambda^{-1}\delta^\dagger\right)^{-1}(Z-Y\Lambda^{-1}\delta^\dagger)^\dagger \\ X & E_m - X\Lambda^{-1}\delta^\dagger & -X\Lambda^{-1}Y^\dagger - (E_m - X\Lambda^{-1}\delta^\dagger)\left(L-\delta\Lambda^{-1}\delta^\dagger\right)^{-1}(Z-Y\Lambda^{-1}\delta^\dagger)^\dagger \\ Y & Z - Y\Lambda^{-1}\delta^\dagger & E_n - Y\Lambda^{-1}Y^\dagger - (Z-Y\Lambda^{-1}\delta^\dagger)\left(L-\delta\Lambda^{-1}\delta^\dagger\right)^{-1}(Z-Y\Lambda^{-1}\delta^\dagger)^\dagger \end{pmatrix}.$$

Next, we treat the time–dependent Hamiltonian like

$$H = H(t) = \begin{pmatrix} H_1(t) & V_1^\dagger(t) & V_2^\dagger(t) \\ V_1(t) & H_2(t) & V_3^\dagger(t) \\ V_2(t) & V_3(t) & H_3(t) \end{pmatrix}. \tag{20}$$

Then the (reduced) evolution equation

$$i\frac{d}{dt}\hat{V} = H(t)\hat{V} \quad \text{where} \quad \hat{V} = \hat{V}(t) \iff X = X(t),\ Y = Y(t),\ Z = Z(t) \tag{21}$$

gives a set of matrix Riccati equations interecting with one another

$$\begin{aligned} i\dot{X} &= V_1 + H_2X - XH_1 - XV_1^\dagger X + V_3^\dagger Y - XV_2^\dagger Y, \\ i\dot{Y} &= V_2 + H_3Y - YH_1 - YV_2^\dagger Y + V_3X - YV_1^\dagger X, \\ i\dot{Z} &= V_3 + H_3Z - ZH_2 - ZV_3^\dagger Z + (ZX-Y)(V_1^\dagger + V_2^\dagger Z). \end{aligned} \tag{22}$$

This is our main result in the paper.

Especially, in the case of $l = m = n = 1$

$$H = \begin{pmatrix} h_1(t) & \bar{v}_1(t) & \bar{v}_2(t) \\ v_1(t) & h_2(t) & \bar{v}_3(t) \\ v_2(t) & v_3(t) & h_3(t) \end{pmatrix} \tag{23}$$

$$\hat{V} = \begin{pmatrix} \frac{1}{\sqrt{\Delta_1}} & \frac{-(\bar{x}+\bar{y}z)}{\sqrt{\Delta_1\Delta_2}} & \frac{\bar{x}\bar{z}-\bar{y}}{\sqrt{\Delta_2}} \\ \frac{x}{\sqrt{\Delta_1}} & \frac{1-(xz-y)\bar{y}}{\sqrt{\Delta_1\Delta_2}} & \frac{-\bar{z}}{\sqrt{\Delta_2}} \\ \frac{y}{\sqrt{\Delta_1}} & \frac{z+\bar{x}(xz-y)}{\sqrt{\Delta_1\Delta_2}} & \frac{1}{\sqrt{\Delta_2}} \end{pmatrix} \tag{24}$$

where

$$\Delta_1 = 1 + |x|^2 + |y|^2, \quad \Delta_2 = 1 + |z|^2 + |xz-y|^2$$

the interacting Riccati equations become

$$
\begin{aligned}
i\dot{x} &= v_1 + (h_2 - h_1)x - \bar{v}_1 x^2 + \bar{v}_3 y - \bar{v}_2 xy, \\
i\dot{y} &= v_2 + (h_3 - h_1)y - \bar{v}_2 y^2 + v_3 x - \bar{v}_1 xy, \qquad (25)\\
i\dot{z} &= v_3 + (h_3 - h_2)z - \bar{v}_3 z^2 + (xz - y)(\bar{v}_1 + \bar{v}_2 z).
\end{aligned}
$$

Concerning these equations, our interest is the following

Problem In this system is there a nonlinear superposition formula like (14) ?

This is an important problem on nonlinear superposition. However, we cannot find such a formula at the present time, so we leave it to readers as a challenging problem.

4 Discussion

In this paper we considered the reduced dynamics from the unitary group $U(N)$ to flag manifolds of the second type and derived a set of differential equations of matrix Riccati types interacting with one another.

We also presented an important problem in the special case ($l = m = n = 1$) on generalization of the nonlinear superposition formula that the Riccati equation satisfies.

In last, let us make a comment. We can generalize our construction to flag manifolds of the third type (in our terminology)

$$F_{k,l,m,n} \cong U(k+l+m+n)/U(k) \times U(l) \times U(m) \times U(n)$$

where

$$F_{k,l,m,n}(\mathbf{C}) = \{\mathcal{U} \subset \mathcal{V} \subset \mathcal{W} \subset \mathbf{C}^{k+l+m+n} \mid \dim_{\mathbf{C}}\mathcal{U} = k,\ \dim_{\mathbf{C}}\mathcal{V} = k+l,\ \dim_{\mathbf{C}}\mathcal{W} = k+l+m\}.$$

In fact, starting from the matrix

$$
F \equiv \begin{pmatrix} E_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ K & E_l & \mathbf{0} & \mathbf{0} \\ L & X & E_m & \mathbf{0} \\ M & Y & Z & E_n \end{pmatrix} \in GL(k+l+m+n)/B_+
$$

we can trace the same process as the text. Explicit calculations done are of course very hard, see [12].

When $k = l = m = n = 1$ some explicit calculations have been done by Picken [10].

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